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On Spaces with Large Chebyshev Subspaces

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This note presents a study of measures on [0, 1] annihilating subspaces $E \subset C[0, 1]$ which contain large Chebyshev subspaces. As an application, we show that every such E is weakly close to a norm-dense subspace. © 1988 Academic Press, Inc.

INTRODUCTION

In this note we deal with subspaces of $C[0, 1]$ that contain Chebyshev systems of arbitrary large dimension. Such subspaces we call C-spaces. We study the properties of functionals that annihilate C-spaces. These functionals and their corresponding measures have very specific properties. In fact, we have the following

Conjecure. Let μ be a Borel measure annihilating a C-space. Then every cluster point of supp μ belongs to the intersection supp $\mu_+ \cap$ supp μ_-.

There is a close connection between this conjecture and Newman's problem [3]. Let $\phi_1, \phi_2, \phi_3, \ldots$ be a sequence of functions in $C[0, 1]$ such that $\{\phi_1, \ldots, \phi_n\}$ span an n-dimensional Chebyshev subspace E_n for every n, and let E be the union of all the E_n, $n \geq 0$. Then E is a C-space. Newman conjectured that the rational functions with numerators and denominators from E are a norm-dense subset of $C[0, 1]$. Using results from [2] we can show that a positive answer to our conjecture would also solve Newman's conjecture.

In this note, we present initial steps towards a possible solution of our conjecture. As an application of our results we study the density of C-spaces in $C[0, 1]$. It is well known that a C-space can be very small (in the sense of Baire category). For example, the subspace generated by all polynomials of the form x^{2^n}, $n \geq 0$, is a C-space which is a nowhere dense subset of $C[0, 1]$. Nevertheless, we show that every C-space is arbitrary close to a dense subspace of $C[0, 1]$. More precisely, we prove

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THEOREM. *There exists a continuous family of bounded operators* T_α, $\alpha > 0$, *such that*

(a) $\lim_{\alpha \to 0} T_\alpha f = f$ *for all* $f \in C[0, 1]$.

(b) $T_\alpha E$ *is a C-space for all C-spaces E and all* $\alpha > 0$.

(c) *For every C-space E and for every* $\alpha > 0$ *the space* $T_\alpha E$ *is norm-dense in* $C[0, 1]$.

THE RESULTS

Let E_n be an $n+1$-dimensional subspace of $C[0, 1]$. If every non-zero $e \in E_n$ has no more than n zeros, then E_n is called a *Chebyshev subspace*. We will use the symbol $Z(e)$ to denote the number (cardinality, resp.) of zeros of functions $e \in C[0, 1]$. The following proposition is well known (cf. [1]).

1. PROPOSITION. *Let* $E_n \subset C[0, 1]$ *be an* $n+1$*-dimensional Chebyshev subspace. Let* $0 = t_0 \leqslant t_1 < t_2 < \cdots < t_m \leqslant t_{m+1} = 1$, $m \leqslant n$. *Then there exists a function* $e \in E_n$ *such that*

$$e(t_j) = 0, \qquad (j = 1, ..., m) \qquad \textit{and} \qquad Z(e) = m$$

and such that $(-1)^i e$ *is positive on* $[t_i, t_{i+1}]$, $0 \leqslant i \leqslant m$.

2. DEFINITION. A (not necessarily closed) subspace $E \subset C[0, 1]$ is called a *space with large Chebyshev subspaces* (C-space) if for every $N \geqslant 0$ there exists $n \geqslant N$ such that E contains an $(n+1)$-dimensional Chebyshev subspace.

In particular, $C[0, 1]$ is a C-space. Moreover, if $\Lambda = \{\lambda_j : j = 0, 1, 2, ...\}$ is an infinite set of real numbers, then

$$E_\Lambda = \text{span}\{1, t^{\lambda_j}: j = 0, 1, 2, ...\}$$

is a C-space. If $\lim_{j \to \infty} \lambda_j = \infty$ and $\sum 1/\lambda_j < \infty$ or if $\lim_{j \to \infty} \lambda_j = 0$ and $\sum \lambda_j < \infty$, then E_Λ is an example of a C-space that is not dense in $C[0, 1]$. Another classical example is span$\{1/(t - \lambda_j); j = 0, 1, 2, ...\}$, provided that $\Lambda \cap [0, 1] = \varnothing$.

We will need some extra terminology from measure theory.

Let $\mathcal{M}[0, 1]$ be the space of all regular Borel measures on $[0, 1]$. As usual, we will identify $\mathcal{M}[0, 1]$ with the dual space of $C[0, 1]$. For every $\mu \in \mathcal{M}[0, 1]$ we write

$$\mu = \mu_+ - \mu_-$$

to denote the usual Hahn decomposition of μ into pairwise orthogonal positive measures. We also define

$$I_\mu = \operatorname{supp} \mu_+ \cap \operatorname{supp} \mu_- .$$

A measure μ is called *peaking* if

$$\mu(f) = \|\mu\| \, \|f\|$$

for some $0 \neq f \in C[0, 1]$. It is well known that μ is peaking if and only if $I_\mu = \emptyset$. If μ is peaking and if $f \in C[0, 1]$ is chosen such that $\mu(f) = \|\mu\| \, \|f\|$, then clearly

$$f_{|\operatorname{supp} \mu_+} = \|f\|, \qquad f_{|\operatorname{supp} \mu_-} = -\|f\|. \tag{1}$$

3. DEFINITION. We will say that $\mu \in \mathcal{M}[0, 1]$ changes its sign k times if there exists a partition of the interval $[0, 1]$

$$0 = t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k = 1$$

such that the restriction measures $\mu_i = \mu_{|(t_i, t_{i+1})}$ are alternatively positive and negative.

4. *Remarks.* (i) If μ is peaking, then the open intervals in Definition 3 can be replaced by the closed intervals $[t_i, t_{i+1}]$.

(ii) If a measure μ is given by a continuous function g, i.e., $\mu(A) = \int_A g(t)\, dt$ for all measurable subsets $A \subset [0, 1]$, then μ changes its sign n times if and only if g changes its signum n times in the sense of continuous functions.

(iii) Clearly not every measure changes its sign only finitely many times. However, if μ is peaking, then μ changes its sign finitely many times.

(Indeed, if $I_\mu = \operatorname{supp} \mu_+ \cap \operatorname{supp} \mu_-$ is empty, then we may cover supp I_+ by finitely many open intervals (x_i, y_i) $1 \leqslant i \leqslant n$; we may assume that the closed intervals $[x_i, y_i]$ do not intersect supp μ_- and are pairwise disjoint. Relabel the x_i such that $x_i < x_{i+1}$. Since the closed intervals are pairwise disjoint, we obtain $y_i < x_{i+1}$. Clearly, since now $[y_i, x_{i+1}] \cap \operatorname{supp} \mu_+ = \emptyset$, the restriction of μ to intervals of the form $[y_i, x_{i+1}]$ yields negative measures. Hence the partition $0 \leqslant x_1 < y_1 < x_2 < y_2 < \cdots < x_n < y_n \leqslant 1$ will do the job.)

The last remark leads immediately to

5. PROPOSITION. *If μ does not change sign only finitely many times, then I_μ is non-empty.*

Using Proposition 1 we obtain

6. PROPOSITION. *Let E_n be an $(n+1)$-dimensional Chebyshev system. Let* $0 \neq \mu \perp E_n$ (i.e., $\mu(e)=0$ *for all* $e \in E$). *If μ changes sign finitely many times, then it changes sign at least $n+2$ times.*

Proof. Suppose that μ changes sgn k times, where $k \leqslant n+1$. Let

$$0=t_0<t_1<\cdots<t_k=1$$

be a corresponding partition. By Proposition 1 there exists $e \in E$ with $e(t_1)=e(t_2)=\cdots=e(t_{k-1})=0$ and $Z(e)=k-1$. Since for each $1 \leqslant i \leqslant k$ the restriction of e to the open interval (t_{i-1}, t_i) is strictly positive, since $e(t_{i-1})=e(t_i)=0$ and since the restriction of μ to (t_{i-1}, t_i) is either positive or negative, we obtain that either $\mu_{|[t_{i-1}, t_i]}=0$ or $\mu_{|[t_{i-1}, t_i]}(e) \neq 0$. Since $0 \neq \mu$, there exists at least one i such that $\mu_{|[t_{i-1}, t_i]} \neq 0$. Hence, if $\mu_{|(t_0, t_1)}$ and $e_{|(t_0, t_1)}$ have always the same sign, then $\mu(e)>0$, otherwise $\mu(e)<0$, contradicting $\mu \perp E$. ∎

Proposition 6 provides us with the following generalization of the Chebyshev theorem.

7. THEOREM. *Let E be a closed subspace of* $C[0,1]$ *that contains an $(n+1)$-dimensional Chebyshev subspace E_n. Let $f \notin E$ and assume there exists $e \in E$ that is a best approximation to f. Then there exist at least $(n+2)$ points $\xi_0, \ldots, \xi_{n+1}$ such that*

$$(f-e)(\xi_j=\lambda \cdot(-1)^j\|f-e\| \qquad \textit{where} \quad \lambda=+1 \quad \text{or} \quad \lambda=-1.$$

Proof. Let e be a best approximation to f. Then by the Hahn–Banach theorem there exists $\mu \perp E$ such that

$$\mu(f-e)=\|\mu\|\;\|f-e\|. \tag{2}$$

Hence μ is a peaking measure and by Proposition 6 it has $k \geqslant (n+2)$ changes of signum. Let $0=t_0<t_1<\cdots<t_k=1$ be the corresponding partition of the unit interval. Choose

$$\xi_i \in \operatorname{supp} \mu_{|[t_i, t_{i+1}]}, \qquad i=0, \ldots, n+1<k.$$

Then by (1) and (2) we have found a collection ξ_i of $n+2$ points satisfying the statement in the theorem. ∎

Actually the proof of the theorem says a little bit more. It describes the set of all $\xi \in [0,1]$ such that $(f-e)(\xi)= \pm\|f-e\|$. They include all the points in supp μ. In particular it follows that if E consists of analytical functions and if f is analytical, then $\{\xi \in [0,1]:(f-e)(\xi)= \pm\|f-e\|\}$ is finite, hence the measure μ in formula (2) is a linear combination of finitely many point evaluations.

We now turn our attention to C-spaces.

8. THEOREM. *Let E be a C-space and let $\mu \perp E$. Then μ does not change sign finitely many times. Therefore, I_μ is non-empty. In partitcular, μ does not peak.*

Proof. For a proof it is sufficient to show that μ does not change sign finitely many times (cf. Proposition 5). Contrary to it, suppose that μ changes sign N times. Then there exists $n \geqslant N$ such that E contains an $(n+1)$-dimensional Chebyshev subspace E_n. Since $\mu \perp E$, μ changes sign at least $n+2 > N$ times by Proposition 6, a contradiction. ∎

It is interesting to mention that while the finite dimensional Chebyshev subspaces are ideally suited for the existence and uniqueness of best approximations, the C-spaces are worst possible.

9. COROLLARY. *Let E be a C-space. Then no $f \notin E$ has a best approximation from E.*

Proof. Suppose that some $f \notin E$ has a best approximation $e \in E$. Then by the Hahn–Banach theorem there exists $\mu \perp E$ such that

$$\mu(f-e) = \|\mu\| \, \|f-e\|.$$

Therefore μ is peaking and thus has finitely many changes of sign by (4.iii), contradicting Theorem 8. ∎

Our next application shows that the C-spaces are in some sense close to being dense in $C[0, 1]$.

10. THEOREM. *Let E be a C-space. Let*

$$T\colon C[0,1] \to C[0,1]$$

be a linear operator or the form

$$T(f)(t) = \int_0^1 k(s,t) f(s) \, ds,$$

where $k(s,t)$ satisfies the following conditions:

(a) *$k(s,t)$ is analytic on $[0,1] \times [0,1]$.*

(b) *$\operatorname{span}\{k(s,-)\colon s \in [0,1]\}$ is dense in $C[0,1]$.*

Then $T(E)$ is dense in $C[0,1]$.

Proof. Let $\mu \perp T(E)$ be a measure annihilating the range of T, i.e., for every element $e \in E$ we have $\mu(T(e)) = 0$. Then $T^*(\mu) \perp E$. Using Fubini's theorem, for the measure $T^*(\mu)$ we have

$$\begin{aligned} T^*(\mu)(f) &= \mu(T(f)) \\ &= \int_0^1 \left[\int_0^1 k(s, t) f(s)\, ds \right] d\mu(t) \\ &= \int_0^1 \left[\int_0^1 k(s, t) f(s)\, d\mu(t) \right] ds \\ &= \int_0^1 f(s) \left[\int_0^1 k(s, t)\, d\mu(t) \right] ds. \end{aligned}$$

This shows that the measure T^* is given by the *analytic* function

$$g(s) = \int_0^1 k(s, t)\, d\mu(t).$$

On the other hand, $T^*(\mu) \perp E$ and therefore by Theorem 8, $T^*(\mu)$ does not change sign finitely many times. Hence $g(s)$ changes sign infinitely many times on $[0, 1]$. It follows that $g(s)$ has infinitely many zeros on the compact interval $[0, 1]$. Hence the analyticity of $g(s)$ implies $g(s) = 0$ for all $s \in [0, 1]$. So we have

$$\int_0^1 k(s, t)\, d\mu(t) = 0 \qquad \text{for all} \quad s \in [0, 1]$$

and thus

$$\mu \perp \operatorname{span}\{k(s, -) : s \in [0, 1]\}.$$

Now implies $\mu = 0$. ∎

This theorem shows that although the closed C-spaces can be small (nowhere dense, for instance), they are weakly close to dense subspaces. Indeed, pick $k_\tau(, t) = g_\tau(t)\, 1/\sqrt{2\pi\tau} \exp(-(s-t)^2/\tau\eta)$, where g_τ, $0 < \tau \leqslant 1$, is a family of analytic functions such that

$$\lim_{\tau \to 0} g_\tau(t) = \begin{cases} 2 & \text{if} \quad t = 0 \text{ or } t = 1, \\ 1 & \text{if} \quad 0 < t < 1. \end{cases}$$

Then the operator T_τ given by k_τ satisfies the conditions of Theorem 10. Hence for all $\tau > 0$ the image of E under T_τ is dense in $C[0, 1]$. Moreover, T_τ converges pointwise to the identity operator as τ tends to zero.

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